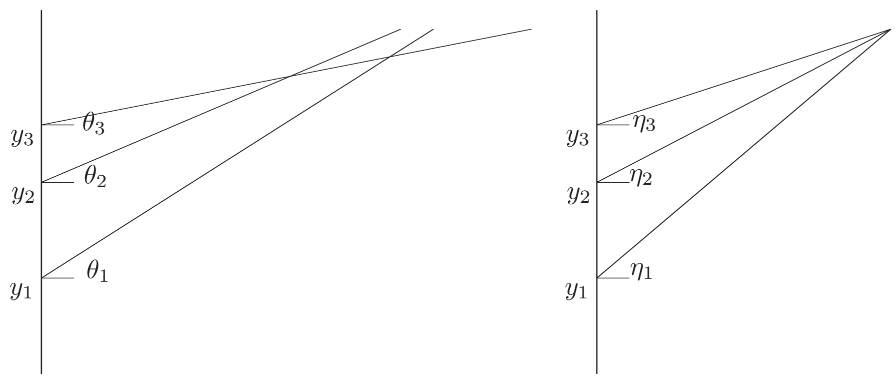
Homework:(Can work as team)
Consider the scenario that three observers, located on a straight line, measure the angle of their line-of-sight to a certain object. Due to various reasons, such as atmosphere turbulence, their observations are obscured.



Assume that the angles are measured counterclockwise from the east (normal to the baseline of observers). It is desired to estimate the true position of the object from three angles θ_i, $i = 1, 2, 3$, which are more or less correct but carry some small uncertainties by minimizing the objective function

$$f(\eta_1, \eta_2, \eta_3) := \sum_{i=1}^{3} (\theta_i - \eta_i)^2, \tag{1}$$

subject to the condition that all three lines of sight intersect at a single point.

1. (20 pts) Show that the least squares problem can be formulated as an equality constrained optimization

$$\begin{array}{ll} \min & f(\eta_1, \eta_2, \eta_3) := \sum_{i=1}^{3} (\theta_i - \eta_i)^2, \\ \text{subject to} & (y_2 - y_1)(\tan\eta_2 - \tan\eta_3) = (y_3 - y_2)(\tan\eta_1 - \tan\eta_2). \end{array}$$

2. (20 pts) Using Lagrange multiplier theory, show that the optimal angles are given by

$$\begin{aligned} \eta_1 &= \theta_1 + \omega(y_2 - y_3)\sec^2\eta_1, \\ \eta_2 &= \theta_2 + \omega(y_3 - y_1)\sec^2\eta_2, \\ \eta_3 &= \theta_3 + \omega(y_1 - y_2)\sec^2\eta_3, \end{aligned}$$

   where ω is a constant that ensures the lines of sight define a single point of intersection.

3. (20 pts) Generalize the problem to n observers at locations $y_1 < y_2 < \ldots < y_n$. Describe the objective function and the constraint(s). (*Hint: How do you ensure that* n *lines will pass through the same point?*)

Computer Project: (Just report the results. No need to follow the paper format, but give sufficient comments.)

1. Generate 21 random points $\{t_k\} \subset [-10, 10]$. Define $y_k := \mathrm{erf}(t_k)$; $k = 1, \ldots, 21$, where $\mathrm{erf}(x)$ is the error function
$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$
Fit the data in a least squares sense with the following curves.

   (a) (25 pts) Fit the data in a least squares sense with polynomials of degrees 1 through 5. Compare the fitted polynomial with $erf(t)$ for values of t between the data points. How does the maximum error depend on the polynomial degree? Make a chart of maximum errors versus n. (*Question to ponder: How would you compute the maximum error?*).

   (b) (15 pts) Because $erf(t)$ is an odd function of t, that is, $erf(x) = -erf(-x)$, it is reasonable to fit the data by a linear combination of odd powers of t. Try degrees 1, 3, and 5. Again, study how the error between data points depends on n. (*Question to ponder: Would this extra care of exploiting oddness help to improve the fitting?*)

   (c) (20 pts) Polynomials are not particularly good approximants for $erf(t)$ because they are unbounded for large t, whereas $erf(t)$ approaches 1 for large t. Try a model of the form $c(t) = c_1 + e^{-t^2}(c_2 + c_3 z + c_4 z^2 + c_5 z^3)$, where $z = 1/(1+t)$. (*How does the error between the data points compare with the polynomial models?)*